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A NOTE ON EXCHANGES IN MATROID BASES

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13. ABSTRACT

In a matroid with bases B and B', a B-exchange is a pair of elements e, e', where B - e + e' is a base. A serial exchange of B into B' is a sequence of pairs e_i, e_i', for i = 1, ..., n, such that e_i, e_i', is a B_{i-1}-exchange, where $B_0 = B$, $B_i = B_{i-1} - e_i + e_i'$, and $B_n = B'$. This paper shows there is a one-to-one correspondence between elements of B and B' such that corresponding elements e, e' give B-exchanges; furthermore, the pairs e, e' can be sequenced to give a serial exchange of B into B'. A symmetric exchange is a pair of elements e, e' such that e, e' is a B-exchange and e', e is a B'-exchange. Any element of B can be symmetrically exchanged with at least one element of B'. But in contrast to B-exchanges, it is not always possible to make a correspondence between B and B' so corresponding elements give symmetric exchanges.

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A NOTE ON EXCHANGES IN MATROID BASES

by

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ABSTRACT

In a matroid with bases B and B', a B-exchange is a pair of elements e, e', where $B - e + e'$ is a base. A serial exchange of B into B' is a sequence of pairs e_i, e_i', for $i = 1, \ldots, n$, such that e_i, e_i' is a B_{i-1}-exchange, where $B_0 = B$, $B_i = B_{i-1} - e_i + e_i'$, and $B_n = B'$. This paper shows there is a one-to-one correspondence between elements of B and B' such that corresponding elements e, e' give B-exchanges; furthermore, the pairs e, e' can be sequenced to give a serial exchange of B into B'. A symmetric exchange is a pair of elements e, e' such that e, e' is a B-exchange and e', e is a B'-exchange. Any element of B can be symmetrically exchanged with at least one element of B'. But in contrast to B-exchanges, it is not always possible to make a correspondence between B and B' so corresponding elements give symmetric exchanges.

Many network and linear programming problems are solved by repeatedly exchanging elements of a base. The pivot step in linear programming is a general example. The existence of such exchanges can be taken as a defining property of a matroid [2]. This note presents results concerning several types of matroid base exchanges.

First we define three types of exchanges. Let M be a matroid, with bases B and B'. For example, Figure shows the graphic matroid on four nodes. One base consists of the solid arcs 1, 2, 3; another base consists of the dotted arcs 4, 5, 6.

An ordered pair of elements e, e' is a <u>B-exchange</u> if $B - \{e\} + \{e'\}$ is a base. Table 1 shows the possible B-exchanges for each element in $B = \{1, 2, 3\}$.

A serial exchange of B into B' is a sequence of ordered pairs, e_1, e_1'; $e_2, e_2', \ldots, e_n, e_n'$, such that for all i in $1 \leq i \leq n$, a base is formed by the set

$$B_i = B - \{e_1, \ldots, e_i\} + \{e_1', \ldots, e_i'\}$$

Furthermore, $B_n = B'$. The definition implies each pair e_i, e_i' is a B_{i-1} exchange. Hence the sequence of exchanges can be executed serially. Figure 2 shows a serial exchange of the base {1, 2, 3} into {4, 5, 6}.

A <u>symmetric exchange</u> is an ordered pair of elements e, e' such that the sets $B - \{e\} + \{e'\}$ and $B' - \{e'\} + \{e\}$ are bases. Equivalently, the pair e, e' is a B-exchange and e', e is a B' exchange. Table 2 shows the possible symmetric exchanges for each element in $B = \{1, 2, 3\}$

To characterize these exchanges, we introduce notation for some well-known matroid concepts [2]. For a base B and an element $f \notin B$, <u>B(f)</u> denotes the unique circuit in the set B + f. In Figure 1, for base $B = \{1, 2, 3\}$, $B(5) = \{2, 3, 5\}$.

For a set of elements D, sp(D) denotes the span of D. This set is defined as the smallest superset of D such that for any element f, if sp(D) + f contains a circuit containing f, then $f \in sp(D)$. In Figure 1, sp ({ 4, 6}) = {2, 4, 6}.

Lemma 1: For elements $e \in B$, $e' \notin B$, these conditions are equivalent:

(i) e, e' is a B-exchange

(ii) $e \in B(e')$.

(iii) $e' \notin sp(B-e)$.

Proof: An immediate consequence of the definitions.

Corollary 1: For elements $e \in B - B'$, $e' \in B' - B$, these conditions are equivalent:

(i) e, e' is a symmetric exchange.

(ii) $e' \in B'(e) - sp(B - e)$

It is apparent from the lemma that any element $e \in B$ gives a B-exchange with at least one element of B'. We show the same is true for symmetric exchanges.

Theorem 1: For any element $e \in B$, there is an element $e' \in B$ such that e, e' is a symmetric exchange.

Proof: Consider any element $e \in B$. If $e \in B'$, then clearly e, e' is a symmetric exchange. So assume $e \notin B'$.

Since B is a base, element $e \notin sp(B-e)$. Thus the circuit B' (e) is not contained in sp(B - e) + e, that is,

$$B'(e) - e \not\subseteq sp(B - e).$$

Now corollary 1 shows there is a symmetric exchange for e, completing the proof.

In Figure 1, we can pair the elements of $B = \{1, 2, 3\}$ and $\{4, 5, 6\}$ so each pair gives a B-exchange: 1, 6; 2, 5; 3, 4. Figure 2 shows these pairs, in the given sequence, are a serial exchange of $\{1, 2, 3\}$ into $\{4, 5, 6\}$. Now we show such a pairing can be made in general.

Theorem 2: There is a one-to-one correspondence between elements of B and B', such that corresponding elements e, e' give a B-exchange. Furthermore, the pairs e, e' can be sequenced to give a serial exchange of B into B'.

Proof: Denote the bases by

$$B = \{e_1, e_2, \ldots, e_n\}, \quad B' = \{e'_1, e'_2, \ldots, e'_n\}.$$

We assert indices can be chosen in B so for all i in $1 \leq i \leq n$, the pair e_i, e'_i is a B-exchange; furthermore, a base is formed by the set

$$B'_i = \{e_1, e_2, \ldots, e_i, \; e'_{i+1}, e'_{i+2}, \ldots, e'_n\}.$$

Note the assertion implies the theorem. For the pairs e_i e'_i give a correspondence of B-exchanges, and the sequence e_n e'_n; e_{n-1}; ...; e_1, e'_1 is a serial exchange of B into B'. The assertion is proved by induction on i. The initial step, $i = 0$, is obvious, since $B'_0 = B$ is a base. For the inductive step, suppose B'_i is a base. We prove the assertion for $i + 1$, as follows. Element e'_{i+1} of base B'_i gives a symmetric exchange with some element of base B. This element cannot be e_j, for j in $1 \leq j \leq i$, since $e_j \in B'_i$. With proper choice of indices, we can assume e_{i+1}, e'_{i+1} is a symmetric exchange. Thus e_{i+1}, e'_{i+1} is a B-exchange, and B'_{i+1} is a base. This completes the induction.

The proof of Theorem 2 gives a constructive procedure for finding the one-to-one correspondence of B-exchanges. The theorem itself, specialized to graphic matroids, is useful in finding minimum weight spanning trees with specified degree at one node [1].

It is natural to try to generalize Theorem 2 to symmetric exchanges. However Table 2 shows it is not always possible to pair the elements of two bases so each pair is a symmetric exchange.

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Figure 1. Bases {1, 2, 3} and {4, 5, 6} in a graphic matroid.

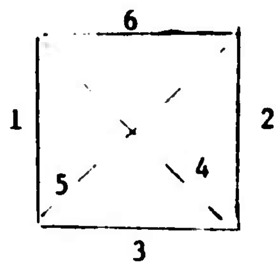


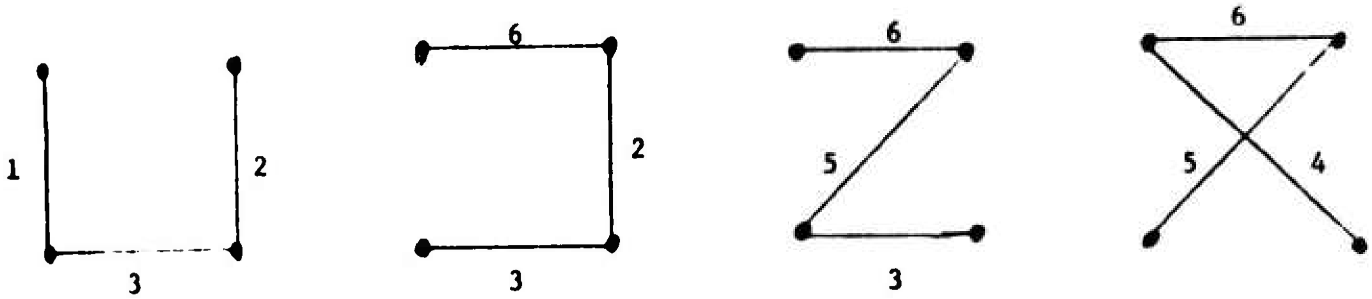
Table 1. B-exchanges for e, B = {1, 2, 3}.

e	e'
1	4, 6
2	5, 6
3	4, 5, 6

Table 2. Symmetric exchanges for e.

e	e'
1	6
2	6
3	4, 5, 6

Figure 2. Serial exchange of {1, 2, 3} into {4, 5, 6}.



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